where R^{-1} is a strictly positive nonlinear scalar function given as follows:

$$\frac{1}{2}R^{-1}(\tilde{x}) = \frac{L_f^* V + \sqrt{\left(L_f^* V\right)^2 + \left(u_{\max}^2 \left(L_{\tilde{g}} V\right)\left(L_{\tilde{g}} V\right)^T\right)^2}}{\left(L_{\tilde{g}} V\right)\left(L_{\tilde{g}} V\right)^T \left[1 + \sqrt{1 + u_{\max}^2 \left(L_{\tilde{g}} V\right)\left(L_{\tilde{g}} V\right)^T}\right]}, \tag{4.28}$$

where $L_{\tilde{g}}V$ is the row vector $[L_{\tilde{g}1}V \cdots L_{\tilde{g}m}V]$ in which V is a quadratic Lyapunov function defined as:

$$V = e^T P e,\ P = \begin{bmatrix} 1 & c' \\ c' & 1 \end{bmatrix},\ c' \in (0,1) \tag{4.29}$$

where P is a positive definite matrix that satisfies the Riccati inequality.[96] Furthermore, $L_f^* V = L_{\tilde{f}} V + \rho |e|^2$ with $\rho > 0$. For practical implementation of the controller given by eqn (4.27), once more a state observer is needed to estimate the state variables that are not measured. For example, an observer of the form (4.18) can be used, which is constructed from the process model including input constraints with the addition of linear gains to account for any differences between the measured and estimated output variables and has the following form:

$$\frac{d\omega}{d\tilde{t}} = \tilde{f}(\omega) + \tilde{g}(\omega)\text{sat}(u) + L\left(y - \tilde{h}(\omega)\right). \tag{4.30}$$

In a case where the process model described by eqn (4.24) obeys a number of basic assumptions (see El-Farra *et al.*[96] for precise definitions), then given a certain initial condition and applicable input constraints, the following inequality should be verified

$$L_f^* V \leq u_{\max} \left| \left(L_{\tilde{g}} V\right)^T \right| \tag{4.31}$$

and if the initial condition is within the region of closed-loop stability, desired closed-loop behaviour can be obtained by using the explicit form of the nonlinear model-based controller as follows:[96]

$$\begin{aligned} \frac{d\omega}{d\tilde{t}} &= \tilde{f}(\omega) - \frac{1}{2}\tilde{g}(\omega) R^{-1}(\omega)\left[L_{\tilde{g}} V(\omega)\right]^T + L\left(y - \tilde{h}(\omega)\right), \\ u &= -\frac{1}{2} R^{-1}(\omega)\left[L_{\tilde{g}} V(\omega)\right]^T. \end{aligned} \tag{4.32}$$

Dynamic simulations of a continuous crystallizer, which included actuator and sensor dynamics, with either a conventional PI controller or the nonlinear model-based controller described by eqn (4.32) have demonstrated that the latter controller has excellent stability properties and is able to drive the output of the system much faster to a desired set point, which is open-loop unstable, compared to when using a conventional PI controller.[96]