of constraints on the process inputs. The hard constraints on the input variables can be modelled using a saturation function as follows:[96]

$$\operatorname{sat}(u_i)=\begin{cases} u_{i,\min} & \text{if } u_i < u_{i,\min} \\ u_i & \text{if } u_{i,\min} \le u_i \le u_{i,\max} \\ u_{i,\max} & \text{if } u_i > u_{i,\max} \end{cases} \tag{4.23}$$

The saturation of input variables limits the attainable steady states of the crystallizer, which is independent of the controller used. Furthermore, the process dynamics may become sluggish or unstable when dynamic trajectories of the input variables are restricted by the saturation function when guiding the system from an initial point to a desired set point. An additional problem during saturation of the input variable is so-called integral "windup". El-Farra *et al.*[96] proposed to use the low-order dynamic model (4.13) of a continuous crystallizer as a basis for a systematic analysis of the attainable set points and robust controller design, which in the presence of input constraints has the following general form:[96]

$$\begin{aligned} \frac{\mathrm{d}\tilde{x}}{\mathrm{d}\tilde{t}} &= \tilde{f}(\tilde{x})+\sum_{i=1}^{m}\tilde{g}_i(\tilde{x})\operatorname{sat}(u_i), \\ \tilde{y}_i &= \tilde{h}_i(\tilde{x}),\ i=1,2,\ldots,m \end{aligned} \tag{4.24}$$

Finding the attainable set points can be done in a straightforward manner by identifying all steady states of the system (4.24) for the set of admissible input variables, *i.e.*,

$$0=\tilde{f}(\tilde{x}_S)+\sum_{i=1}^{m}\tilde{g}_i(\tilde{x}_S)u_i^*,\ \text{with: } u_i^*\in\left[u_{i,\min},u_{i,\max}\right], \tag{4.25}$$

which can be obtained, for example, by a nonlinear algebraic solver in combination with a continuation algorithm. The set of attainable set points for each controlled variable follows directly from the computed state vectors at steady state:

$$v_i=\tilde{h}_i(\tilde{x}_S),\ i=1,2,\ldots,m. \tag{4.26}$$

A practical benefit of identifying the attainable set points in the presence of input constraints is that an operator can know in advance the range of set points that can be achieved irrespective of the type of controller to be used. If a set point is attainable, the next question is how to design a robust controller that can track a set point in the presence of the input constraints. El-Farra *et al.*[96] proposed the use of Lyapunov-based control methods to synthesize such a feedback controller. In particular, they proposed a nonlinear feedback controller that has certain optimality properties in its ability to accomplish the control objectives using small control actions of the following form:

$$u=-\frac{1}{2}R^{-1}(\tilde{x})\left(L_{\tilde{g}}V\right)^T, \tag{4.27}$$