but this is known to lead to numerical diffusion – an artificial broadening – of the CSD. The second order term – third term in the right hand side of eqn (2.63) – is aimed to reduce the numerical diffusion, but, this term leads to numerical oscillations around the sharp fronts of the CSD. The flux limiter function (ϕ_l) adaptively turns the second order term off around the sharp CSD variations, avoiding the numerical oscillation, and keeping the second order term active in the rest of the solution to reduce numerical diffusion. The FVM combined with flux limiter function is also known as the high resolution finite volume method (HR FVM) due to its ability to capture sharp fronts or discontinuity without numerical oscillation. $\phi_l = f(\theta_l)$ is calculated from the ratio of consecutive gradients (θ_l):

$$\theta_l = \frac{f_l^m - f_{l-1}^m}{f_{l+1}^m - f_l^m} \tag{2.64}$$

Although numerous flux limiter functions were developed for different finite volume schemes, for crystallization the Van Leer flux limiter is the most widely used (Gunawan *et al.* 2004):

$$\phi(\theta_l) = \frac{|\theta_l| + \theta_l}{1 + |\theta_l|} \tag{2.65}$$

The presented FVM is a discrete time formulation, which means that in addition to the crystal size, the time is also discretized. In these schemes the time step is recalculated in every iteration to satisfy the Courant–Friedrichs–Lewy (CFL) criterion. The numerical system is stable if CFL ≤ 1.

$$\mathrm{CFL} = \frac{k}{h} \max\{G_l\} \tag{2.66}$$

Practically, the CFL is fixed and k is expressed from eqn (2.66).

The FVM methods are generic PBE solvers as they are able to provide full CSD. The implementation of nucleation, growth and dissolution is straightforward. FVM was successfully applied for the numerical solution of aggregation-breakage problems. On the other hand, the computational expense of the FVM is significantly higher than that of the moment based methods.

2.6 Advanced Crystallization Modeling – Case Studies

In this section, case studies will be discussed for topics that are beyond the standard application of the PBMs in pharmaceutical crystallization process, without the aim of providing a comprehensive description of all possible application areas. The basic PBEs with nucleation and growth will be used for demonstration. Naturally, these PBMs can be combined (extended) with the models presented so far, in terms of crystallization mechanisms, crystallizer type and numerical solution techniques.